STOCHASTIC MODELS OF HEAT AND NUCLEAR PARTICLE TRANSFER BASED ON GENERALIZED EQUATION OF FOKKER-PLANCK-KOLMOGOROV

by

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Stochastic differential equations are proposed on the basis of generalized Fokker-Planck--Kolmogorov equation. From the statements of boundary and initial value problems for probabilistic density, dispersion and differential entropy the conditions for stability of solutions and changes in scenarios of development of random phenomena of heat and nuclear particle transfer are obtained.

Key words: stochastic differential equation, dispersion, stability of the solution

INTRODUCTION

Stochastic differential equations are used to model many situations including population dynamics, protein kinetics, turbulence, finance, and engineering. Knowing the solution of the stochastic differential equations in question leads to interesting analysis of the trajectories. Most stochastic differential equations are unsolvable analytically and other methods must be used to analyze properties of the stochastic process [1]. From the stochastic differential equation, a partial differential equation can be derived to give information on the probability transition function of the stochastic process. Knowing the transition function gives information on the equilibrium distribution (if one exists), and convergence to the equilibrium distribution. By Newton's second law, the movement of a Brownian particle can be described by the differential equation, called the Langev in equation, given by $m\mathrm{d}^2x/\mathrm{d}t^2 \quad F(x,t)$, where the force $F(x, t)$, is the sum of a deterministic and random forces. Thus, the position of the particle at a time t, $x(t)$, is a stochastic process and our goal is to understand the transition probabilities in this model.

In this work the problems of stochastic description of deterministic mathematical models of heat transfer are discussed. Especially important is the application of stochastic differential equations in solving transport of nuclear particles (charged and uncharged) through the material. This problem is actualized in recent times coupled with the increasing trend towards miniaturization of electronic components, which increases their reliability in the field of particle radiation [2-4].

The fundamental position is as follows: a solution of a deterministic problem is the expected value of its stochastic analogue. The relevance of the problem is in the fact that deterministic problem neither reveals the stability of stochastic phenomena nor the change of scenarios of its time development. We propose the method of investigation of solutions stability of differential equations for the expected values of temperature based on the analysis of dispersion. Also, the change of scenarios of development of random phenomena is discussed supported by the investigation of differential entropy.

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ANALYTICAL METHOD AND RESULTS

Classical Fokker-Planck-Kolmogorov equation [5-7], is written without the term which is responsible for spasmodic behavior of random phenomena which has the following form

$$\frac{\partial P(t,\Omega)}{\partial t} \qquad \frac{\partial[A(t,\Omega)P(t,\Omega)]}{\partial\Omega} \quad 0.5B\,\frac{\partial^2 P(t,\Omega)}{\partial\Omega^2} \tag{1}$$

Here, $P(t, \Omega)$ is probability density function (in what follows: PB), $A(t, \Omega)$ is drift coefficient, and B is Markov diffusion coefficient where t is a time co-ordinate and Ω is a characteristic of random field.

The basis for obtaining stochastic analogues of problems corresponding to deterministic problems of heat transfer is the equation, proposed in [8], which generalizes Fokker-Planck-Kolmogorov equation for the function of probability density $P(t, x, \Omega)$, where x is a space co-ordinate, written without a term responsible for spasmodicity. It has the following form for description of phenomena of heat conductivity

$$\frac{\partial P(t,x,\Omega)}{\partial t} = a\frac{\partial^2 P(t,x,\Omega)}{\partial x^2} - \frac{\partial[A(t,x,\Omega)P(t,x,\Omega)]}{\partial \Omega} + 0.5B\frac{\partial^2 P(t,x,\Omega)}{\partial \Omega^2} \quad (2)$$

Here, a is the coefficient of the temperature conductivity.

The expected value of the temperature and the second moment are

$$M^{(1)}(t,x) = \int_{-\infty}^{\infty} \Omega P(t,x,\Omega)\mathrm{d}\Omega, \quad M^{(2)}(t,x) = \int_{-\infty}^{\infty} \Omega^2 P(t,x,\Omega)\mathrm{d}\Omega \quad (3)$$

and dispersion is denoted by

$$\sigma^2(t,x) = M^{(2)}(t,x) - [M^{(1)}(t,x)]^2 \quad (4)$$

Differential entropy is introduced in the following way

$$S(t,x) = -\int_{-\infty}^{\infty} P(t,x,\Omega)\ln P(t,x,\Omega)\mathrm{d}\Omega \quad (5)$$

Let us describe the basic facts of the phenomenological modeling of random phenomena. The following discussion differs from the method of "local expected values" [8, 9], but it also leads to the given equation.

We consider the classical equation of heat conductivity as an equation for the expected values

$$\frac{\partial \int_{-\infty}^{\infty} \Omega P(t,x,\Omega)\mathrm{d}\Omega}{\partial t} - a\frac{\partial^2 \int_{-\infty}^{\infty} \Omega P(t,x,\Omega)\mathrm{d}\Omega}{\partial x^2} = 0 \quad (6)$$

If we re-write this equation in the following form

$$\int_{-\infty}^{\infty} \Omega \left[\frac{\partial P(t,x,\Omega)}{\partial t} - a\frac{\partial^2 P(t,x,\Omega)}{\partial x^2}\right]\mathrm{d}\Omega = 0 \quad (7)$$

we obtain

$$\frac{\partial P(t,x,\Omega)}{\partial t} - a\frac{\partial^2 P(t,x,\Omega)}{\partial x^2} = 0 \quad (8)$$

In this equation there are no terms responsible for exterior random influence. Since the principle of correspondence has to hold, the equation for *PB* with the fixed space co-ordinate and zero value of coefficient of temperature conductivity has to reduce to the classical Fokker-Planck-Kolmogorov equation, which on the right hand side contains the component with the drift coefficient and the component with the Markov diffusion coefficient. Consequently, in order to respect the principle of correspondence, the equation for *PB* has to be written in the following form

$$\frac{\partial P(t,x,\Omega)}{\partial t} = a\frac{\partial^2 P(t,x,\Omega)}{\partial x^2} - \frac{\partial[A(t,x,\Omega)P(t,x,\Omega)]}{\partial \Omega} + 0.5B\frac{\partial^2 P(t,x,\Omega)}{\partial \Omega^2} \quad (9)$$

Let us prove, that under the uniqueness conditions

$$P(0,x,\Omega) = P_{\mathrm{init}}(x,\Omega),\ x \in [0,l]$$
$$P(t,0,\Omega) = P_0(t,\Omega),\ P(t,l,\Omega) = P_1(t,\Omega),$$
$$t \in [0,\infty),\ \Omega \in (-\infty,\infty) \quad (10)$$

for which the following equations are satisfied

$$\int_{-\infty}^{\infty} P_{\mathrm{init}}(x,\Omega)\mathrm{d}\Omega = 1,\ x \in [0,l]$$
$$\int_{-\infty}^{\infty} P_0(t,\Omega)\mathrm{d}\Omega = 1,\ \int_{-\infty}^{\infty} P_l(t,\Omega)\mathrm{d}\Omega = 1,\ t \in [0,\infty]$$
$$P_{\mathrm{init}}(x,\pm\infty) = P_0(t,\pm\infty) = P_l(t,\pm\infty) = 0, \quad (11)$$
$$x \in [0,l],\ t \in [0,\infty)$$

the condition of normalization of *PB* (to the unit) holds. We obtain

$$\frac{\partial \int_{-\infty}^{\infty} P(t,x,\Omega)\mathrm{d}\Omega}{\partial t} - a\frac{\partial^2 \int_{-\infty}^{\infty} P(t,x,\Omega)\mathrm{d}\Omega}{\partial x^2} = \int_{-\infty}^{\infty}\left[-\frac{\partial[A(t,x,\Omega)P(t,x,\Omega)]}{\partial \Omega} + 0.5B\frac{\partial^2 P(t,x,\Omega)}{\partial \Omega^2}\right]\mathrm{d}\Omega \quad (12)$$

Since *PB* and its derivatives are zero at infinity, we obtain that the right hand side of the last equation is zero

$$\frac{\partial \int_{-\infty}^{\infty} P(t,x,\Omega)\mathrm{d}\Omega}{\partial t} - a\frac{\partial^2 \int_{-\infty}^{\infty} P(t,x,\Omega)\mathrm{d}\Omega}{\partial x^2} = 0 \quad (13)$$

and get a problem for a parabolic equation with unitary uniqueness conditions. From the uniqueness of the solution for such a problem, we conclude that $\int_{-\infty}^{\infty} P(t,x,\Omega)\mathrm{d}\Omega = 1$ for all possible space and time co-ordinates and so the normalization condition is satisfied.

Let us also note that all functions in the uniqueness conditions for *PB* are nonnegative. Owing to the principle of maximum for a parabolic equation we conclude that for all possible values of space and time coordinates the *PB* is nonnegative.

If we derive the time dependence of entropy on dispersion in the case of normal distribution, we calculate

$$S(t,x) \quad \overset{\infty}{\underset{\infty}{}} P(t,x,\Omega)\ln P(t,x,\Omega)\mathrm{d}\Omega$$

$$\overset{\infty}{\underset{\infty}{}} \frac{1}{\sigma(t)\sqrt{2\pi}}\exp \quad \frac{[\Omega \quad M_T^{(1)}(t,x)]^2}{2\sigma^2(t)}$$

$$\ln \quad \frac{1}{\sigma(t)\sqrt{2\pi}}\exp \quad \frac{[\Omega \quad M_T^{(1)}(t,x)]^2}{2\sigma^2(t)} \quad \mathrm{d}\Omega$$

$$\ln\sqrt{2\pi}\sigma(t) \quad \frac{\sigma^2(t)}{2\sigma^2} \quad \sigma(t)\ln\sqrt{2\pi} \quad \frac{1}{2} \tag{14}$$

TEMPERATURE IN A ROD

Let us consider the problem for the description of stochastic behavior of temperature in a rod at the moment of time $0 \quad t \quad t_1$. We have [7]

$$\begin{gathered}\frac{\partial M_T^{(1)}}{\partial t} \quad a\frac{\partial^2 M_T^{(1)}}{\partial x^2}, \quad 0 \quad t \quad t_1, \quad 0 \quad x \quad l \\ M_T^{(1)}(t,0) \quad M_{T0}^{(1)} \quad 0, \quad 0 \quad t \quad t_1 \\ M_T^{(1)}(t,l) \quad M_{T0}^{(1)} \quad 0, \quad 0 \quad t \quad t_1 \\ M_T^{(1)}(0,x) \quad M_{T0}^{(1)} \quad 0, \quad 0 \quad x \quad l\end{gathered} \tag{15}$$

Stochastic mode of the behavior of rode temperature at moments of time $0 \quad t \quad t_1$ is described by the normal distribution law [7]

$$P(t,x,\Omega) \quad \frac{1}{\sigma_1(t)\sqrt{2\pi}}\exp \quad \frac{[\Omega \quad M_{T0}^{(1)}(t,x)]^2}{2\sigma_1^2(t)} \tag{16}$$

Here, $\sigma_1(t)$ = const $\quad 0$ and the entropy $S(t)$ = $= \sigma_1(t)\ \ln(2\pi)^{1/2} \quad 1/2$. Temperature on the boundary and within a rod up to the moment t_1 is constant and it is equal to the initial temperature $M_T^{(1)}(t) \quad M_{T0}^{(1)}$ $=$ const $\quad 0, 0 \quad t \quad t_1, 0 \quad x \quad l.$

Starting from the moment of time t_1, the dispersion experiences a jump and it is equal to $\sigma_2^2 \quad$ const $\quad \sigma_1^2$. Statement of the problem for expected values of temperature at moments of time $t_1 \quad t \quad t_2$ has the following form [7]

$$\begin{gathered}\frac{\partial M_T^{(1)}}{\partial t} \quad a\frac{\partial^2 M_T^{(1)}}{\partial x^2}, \quad t_1 \quad t \quad t_2, \quad 0 \quad x \quad l \\ M_T^{(1)}(t,0) \quad M_{T0}^{(1)} \quad 0, \quad t_1 \quad t \quad t_2 \\ M_T^{(1)}(t,l) \quad M_{T0}^{(1)} \quad 0, \quad t_1 \quad t \quad t_2 \\ M_T^{(1)}(0,x) \quad M_{T0}^{(1)} \quad 0, \quad 0 \quad x \quad l\end{gathered} \tag{17}$$

Also, we suppose that stochastic mode of temperature behavior at the boundaries and inside the region at moments of time $t_1 \quad t < t_2$ is described by normal distribution

$$\begin{gathered}P(t,x,\Omega) \quad \frac{1}{\sigma_2(t)\sqrt{2\pi}}\exp \quad \frac{[\Omega \quad M_{T0}^{(1)}(t,x)]^2}{2\sigma_2^2(t)}, \\ \sigma_2(t) \quad \text{const.}\end{gathered} \tag{18}$$

Differential entropy, as well as dispersion at the moment t_2, experiences a jump, which foretells soon change of the behavior of temperature of the rod. In the meantime, at moments of time $t_1 \quad t < t_2$ temperature on the boundary and inside the rod is constant and equal to $M_T^{(1)}(t) \quad M_{T0}^{(1)} \quad$ const $\quad 0, t_1 \quad t \quad t_2, 0 \quad x \quad l$, *i. e.*, it is the same as it was till t_2.

Suppose that, starting from the moment $t_3 > t_2$, the dispersion experiences a new jump and increases with time, for example, by the linear law $\sigma_3^2(t) \quad \sigma_{30}^2 \quad \alpha(t \quad t_2)$. Here $\sigma_{30}^2 \quad const \quad \sigma_2^2$, and let the stochastic mode of temperature behavior at the boundary of the rod $x = l$ at the moment $t_3 > t_2$ is also given by the normal distribution

$$P(t,l,\Omega) \quad \frac{1}{\sigma_3(t)\sqrt{2\pi}}\exp \quad \frac{[\Omega \quad M_{T0}^{(1)}(t,l)]^2}{2\sigma_2^2(t)} \tag{19}$$

We consider the case when, starting from this moment of time t_3, the statement of the problem for expected values is changed, for example, as follows

$$\begin{gathered}\frac{\partial M_T^{(1)}}{\partial t} \quad a\frac{\partial^2 M_T^{(1)}}{\partial x^2}, \quad t_3 \quad t \quad \infty, \quad 0 \quad x \quad l \\ M_T^{(1)}(t,0) \quad M_{T0}^{(1)} \quad 0, \quad t_3 \quad t \quad \infty \\ M_T^{(1)}(t,l) \quad 0, \quad t_3 \quad t \quad \infty \\ M_T^{(1)}(t_3,x) \quad M_{T0}^{(1)} \quad 0, \quad 0 \quad x \quad l\end{gathered} \tag{20}$$

Here, the boundary condition for $x = l$ has changed. Differential entropy $S(t) \quad \sigma_3(t)\ln(2\pi)^{1/2} \quad 1/2$ at moment t_3 experiences a new jump, representing a major behavior of temperature field. Temperature has stopped being constant and started to change by the law

$$\begin{gathered}M_T^{(1)}(t,x) \\ M_{T0}^{(1)} \sum_{n \quad \infty}^{\infty} 1 \quad \Phi\left|x \quad \frac{2nl}{2a\sqrt{t \quad t_2}}\right| \operatorname{sign}(x \quad 2nl)\end{gathered} \tag{21}$$

where Φ is the error function.

CONCLUSION

This analysis shows that under the given uniqueness conditions the normalization of probability density holds and it is not possible to predict change of scenarios of behavior of random phenomena only by following the expected values. This change follows in the result of the analysis of the emergence of jumps of dispersion and differential entropy at the boundary of the region.

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AUTHORS' CONTRIBUTIONS

Both authors have contributed to model analysis and results, theoretical analysis and literature research. The manuscript was prepared and written by D. Ć. Dolićanin-Djekić.

REFERENCES

[1] Hottovy, S., The Fokker-Planck Equation, University of Wisconsin, Department of Mathematics, Madison, Wis., USA, 2011
[2] Stanković, K. D., Influence of the Plain-Parallel Electrode Surface Dimensions on the Type a Measurement Uncertainty of GM Counter, *Nucl Technol Radiat, 26* (2011), 1, pp. 39-44
[3] Kovačević, A. M., *et al*., The Combined Method for Uncertainty Evaluation in Electromagnetic Radiation Measurement, *Nucl Technol Radiat, 29* (2014), 4, pp. 279-284
[4] Osmokrović, P., *et al.*, Reliability of Three-Electrode Spark Gaps, *Plasma Devices and Operations, 16* (2008), 4, pp. 235-245
[5] Shapovalov, A. V., *et al*., Symmetry Operators for the Fokker-Plank-Kolmogorov Equation with Nonlocal Quadratic Nonlinearity, Symmetry, Integrability and Geometry: *Methods and Applications*, *3* (2007), 005, p. 16
[6] Trifonov, A. Yu., Trifonova, L. B., Fokker-Planck-Kolmogorov Equations with a Nonlocal Nonlinearity in a Semi Classical Approximation, *Izv. Vyssh. Uchebn. Zaved. Fiz., 45* (2002), (English transl.: *Russian Phys. J., 45* (2002)), pp. 118-128
[7] Soloviev, I. A., *et al*., The Stochastic Approach to Research of Stability of Solutions of Differential Equations, State University of Novi Pazar, Novi Pazar, Serbia, 2014
[8] Soloviev, I. A., The Equations for the Random Heat Fields (in Russian), *Inzhenerno-Fizicheskii Zhurnal, 73* (2000), 2, pp. 396-400
[9] Soloviev, I. A., The Equations for Stochastic Fields, State University of Land Use Planning, Moscow, 2006
[10] Frank, T. D., Nonlinear Fokker-Planck Equations, Springer, Berlin, 2004
[11] Ovsjannikov, L. V., Group Analysis of Differential Equations (in Russian), Nauka, Moscow, 1978 (English transl.: *Academic Press*, New York, 1982)
[12] Schuller, F. P., Vogt, P., Product Structure of Heat Phase Space and Branching Brownianmotion, *Ann. Phys., 308* (2003), 528554
[13] Bezverbny, A. V., *et al*., Nonlinear Fokker-Planck-Kolmogorov Equation in the Semi Classical Coherent Trajectory Approximation (in Russian), *Izv. Vyssh. Uchebn. Zaved. Fiz., 48* (2005), (English transl.: *Russian Phys. J., 48* (2005), 6, pp. 592-604

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СТОХАСТИЧКИ МОДЕЛИ ПРЕНОСА ТОПЛОТЕ И НУКЛЕАРНИХ ЧЕСТИЦА БАЗИРАНИХ НА ГЕНЕРАЛИЗОВАНОЈ ЈЕДНАЧИНИ ФОКЕР-ПЛАНК-КОЛМОГОРОВА

Стохастичке диференцијалне једначине предложене су на бази генерализоване једначине Фокер-Планк-Колмогорова. Из граничних и почетних услова проблема густине вероватноће, дисперзије и диференцијалне ентропије добијени су услови стабилности решења и промене у сценарију развоја случајних појава преноса топлотних и нуклеарних честица.

Кључне речи: стохастичка диференцијална једначина, дисперзија, стабилност решења