A MODERN MATHEMATICAL METHOD FOR FILTERING NOISE IN LOW COUNT EXPERIMENTS

by

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In the proposed work, a novel application of a numerical and functional analysis based on the discrete wavelet transform is discussed. The mathematics of improving signals and removing noises are described. Results obtained show that the method used in a variety of gamma spectra is superior to other techniques.

Key words: noise, signal, discrete wavelet transform

INTRODUCTION

Most experiments in gamma ray and activation spectroscopy require high resolution accompanied by low electronic noise. Germanium and scintillation spectrometers, used independently or in coincidence and coupled with photomultipliers, are applied to obtain a significant reduction of fluctuations. The detected spectra are often accompanied by noise due to many reasons, resulting in random, spurious fluctuations of the signal received at the detector. It is, therefore, advisable to find a simple method for smoothing the signals and removing noises, especially in low count experiments [1-5].

The discrete wavelet transform (DWT) is considered a simple and accurate method which can be used for improving the detected spectra, especially in low-background experiments. The mathematical theory for DWT dates back to Fourier's theory of 1822 for decomposing signals according to their frequencies. DWT may decompose a signal directly according to the frequency and transform it from the time domain to the frequency domain. In the transformation, both time and frequency information of the signal are retained. Mathematicians moved from the concept of frequency analysis to that of scale analysis by creating a function that is shifted by some translation and scaled. This process can be repeated by new shifts and scales of the previous structures. At each step, a new approximation of the signal can be accomplished [6].

Several methods have been applied for smoothing and removing noise from the signals, but a distinction is seldom made between the procedures. Smoothing removes components (of the transformed signal) occurring at the high end of the transformed domain regardless of amplitude, while denoising removes all amplitude components occurring in the transformed domain, regardless of their position. Thus, DWT can decompose a signal into several scales that represent different frequency bands and at each scale, the position of the signal's instantaneous structure can be determined approximately [7, 8].

In this work, DWT is applied for smoothing and denoising low-count background spectra detected by germanium and scintillation spectrometers. The DWT is compared with other known methods of smoothing and denoising.

THEORETICAL BASIS

Historical background

The mathematical theory of DWT stems from Fourier's theory for decomposing signals according to their frequencies. DWT may decompose a signal directly according to frequency and transform it from the time domain to the frequency domain [9]. In the transformation, both time and frequency information of the signal are retained. Mathematicians moved from the concept of frequency analysis to that of scale analysis in which the functions created are shifted by translation and scaled with different scaling functions. The process can be repeated by new shifts and scales of the previous structures. At each step, a new approximation of the signal can be accomplished. The decompositions of a signal using DWT occur at different scales and positions. So, all basis functions are $\Psi_{a,b}(t)$ de-

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rived from a mother wavelet through the following dilation and translation processes [10]

$$\Psi_{\mathrm{a,b}}(t) = \frac{1}{\sqrt{a}}\Psi\left(\frac{t-b}{a}\right) \quad a,b \in R, a \neq 0 \tag{1}$$

where a and b are, respectively, scale and location parameters, with $a > 0$ and b having arbitrary values. The mother wavelet $\Psi(t)$ is chosen to serve as a prototype for all basis functions in the process. All basis functions used are the dilated (or compressed) and shifted versions of the mother wavelet. A number of different functions were applied for this purpose.

Discrete wavelet transforms

Since spectra detected by germanium and scintillation spectrometers are Gaussian-shaped, a mother wavelet of the wavelet family of Gaussian functions, such as the Marr wavelet or the Mexican hat, would be a good choice. The continuous DWT of a signal $X(t)$ is given by [11]

$$T_{\mathrm{a,b}} = \int_{-\infty}^{\infty} X(t)\frac{1}{\sqrt{a}}\Psi\left(\frac{t-b}{a}\right)\mathrm{d}t \tag{2}$$

In practical computation, since the spectrum to be analyzed is often discrete sampling data, the discrete form of the wavelet is preferred. In the discrete wavelet transform treatment eq. (1) is represented as

$$\Psi_{\mathrm{m,n}}(t) = \frac{1}{\sqrt{a_0^m}}\Psi\left(\frac{t - nb_0a_0^m}{a_0^m}\right) \tag{3}$$

where m and n control the wavelet dilation and translation, respectively. The DWT treatment of the signal $X(t)$ will be

$$T_{\mathrm{(m,n)}} = \int_{-\infty}^{\infty} X(t)\frac{1}{\sqrt{a_0^m}}\Psi(a_0^{-m}t - nb_0)\mathrm{d}t \tag{4}$$

where, $T_{\mathrm{m,n}}$ represent the DWT values given on a scale -location grid of the m, n index. The choice of $a_0 = 2$ and $b_0 = 1$ is known as the dyadic grid arrangement. Usually, the discrete dyadic grid wavelets of the scaling functions must be orthogonal to their discrete translations. This means the information stored in a wavelet is not repeated elsewhere and allows for the complete regeneration of the original signal without redundancy. Dyadic discrete wavelets are associated with the scaling function and dilation equations. The scaling function is associated with the smoothing of the signal and has the same form as the wavelet, as given by

$$\phi_{\mathrm{m,n}}(t) = 2^{-m/2}\phi(2^{-m}t - n) \tag{5}$$

So, the scaling function can be convoluted with the signal to produce approximation coefficients as follows

$$S_{\mathrm{m,n}} = \int_{-\infty}^{\infty} x(t)\phi_{\mathrm{m,n}}(t)\mathrm{d}t \tag{6}$$

Now the signal $X(t)$ can represent a combination of an approximation coefficient and the detailed coefficients as

$$X(t) = \sum_{n=-\infty}^{\infty} S_{\mathrm{m,n}}\phi_{\mathrm{m,n}}(t) + \sum_{m=-\infty}^{m}\sum_{n=-\infty}^{\infty} T_{\mathrm{m,n}}\psi_{\mathrm{m,n}}(t) \tag{7}$$

The wavelet basis is a set of functions which are defined by a recursive difference eq

$$\phi(t) = \sum_k c_{\mathrm{k}}\phi(2t - k) \tag{8}$$

where $\phi(2t-k)$ is a contracted version of $\phi(t)$ translated along the time axis by an integer step k and factored by an associated scaling coefficient, c_{k}. The value of the coefficients is determined by the constraints of orthogonality and normalization which require

$$\sum_k c_{\mathrm{k}} = 2 \tag{9}$$

From the orthogonality conditions, scaled functions can be derived as

$$\psi(t) = \sum_k (-1)^k c_{1-k}\phi(2t - k) \tag{10}$$

which dependens upon the solution of $\psi(t)$. Normalization requires that

$$\sum_k c_k c_{k+2m} = 2\delta_{0m} \tag{11}$$

meaning that the above sum is zero for all m not equal to zero and that the sum of the squares of all coefficients is two. Another important equation which can be derived from the above conditions and equations is

$$\sum_k (-1)^k c_{1-k} c_{k+2m} = 0 \tag{12}$$

A good way to solve these equations is to construct a matrix of coefficient values.This is a square M × M matrix where M is the number of non-zero coefficients. The matrix always has an eigenvalue equal to 1. Once these values are known, all other values of the function can be generated by applying the recursion equation to obtain the desired dilation.

Spectrum denoising algorithm

Denoising of spectra depends on a common routine known as the pyramid algorithm. It is an efficient method, especially in spectra detected by HPGe and scintillation detectors. The algorithm operates on a finite set of 2^n input data, where n is an integer. The data are passed through two filters which create an output stream that is half the length of the original input. The filters are one half of the output produced by the low-pass filter function, related to eq. (8)

$$a_{\mathrm{i}} = \frac{1}{2}\sum_{j=1}^{N} c_{2i-j+1}f_j \quad i = 1, \ldots, \tfrac{N}{2} \tag{13}$$

the other half is produced by the high-pass filter function, related to eq. (10)

$$b_i = \frac{1}{2}\sum_{j=1}^{N}(-1)^{j+1}c_{j+2-2i}f_j \quad i = \ldots,\frac{N}{2} \tag{14}$$

where N is the total channel number of the spectrum, c are the coefficients, f is the input function, and a and b are the output functions. The low and high-pass outputs are usually referred to as the odd and even outputs, respectively. The low-pass output contains most of the information of the original input signal. The high-pass output contains the difference between the true input and the value of the reconstructed one.

An important step in denoising signals is finding wavelet functions which cause the even terms to be nearly zero, such as the Haar wavelet which represents a simple interpolation scheme.

After passing these spectrum data through the filter functions, the high-pass filter obviously contains less information than the low-pass output. Since the perfect reconstruction is a sum of the inverse low-pass and inverse high-pass filters, it is necessary to calculate both of them as

$$f_j^L = \sum_{j=1}^{N/2} c_{2i-j}a_i, \; j = 1,\ldots,N \tag{15}$$

$$f_j^H = \sum_{j=1}^{N/2}(-1)^{j+1}c_{j+1-2i}b_i, \; j = 1,\ldots,N \tag{16}$$

So, the reconstruction of the spectrum is expressed as

$$f = f_j^L + f_j^H \tag{17}$$

Since most of the information exists in the low-pass filter output, one can envision removing the noise by taking the filter's output and transforming it again to get two new sets of data, each a quarter of the size of the original input. If, again, little noise is carried by the high-pass output, it can be discarded. Each step of re-transforming the low-pass output, the so-called decomposition, reaches a maximum n numbers, while the total channel number of the input spectrum is $N = 2^n$. The whole process of decompositions is shown in fig. 1. The general procedure of denoising and smoothing is summarized as follows: (1) applying the DWT to a noisy spectrum and obtaining DWT coefficients, (2) removing the noise by deleting the coefficients associated with the noise, and (3) reconstruction of the signal to obtain the signal after the removal of the noise.

MATERIAL AND METHODS

Simulated spectra

Simulated spectra with different single and double peaks were generated. Correlated noise was added to the spectrum using random generators. Random numbers were produced using a mathematical approach. Basically, thus was done by starting with a number called the seed and by then applying a mathematical formula which produces another number. The quality of the random numbers generated depends on the length of the cycle and a 1.0 chance of each number of the set to become a selected number, as well as the extent to which the 1st, 2nd, 3rd... *etc.*, number selected after the each selected one is supposedly unrelated to previous numbers [12].

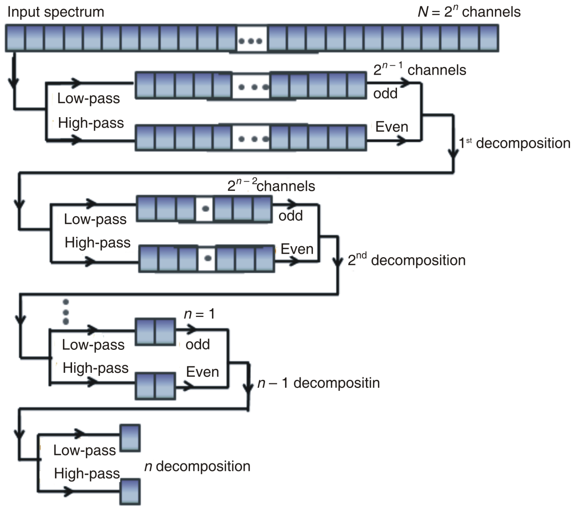

Figure 1. Block diagram of spectrum decomposition using the discrete wavelet transform

Filter optimization

To determine the most appropriate wavelet filters for removing noises from the spectra, 11 different filters from different families, Haar (H), Coiflet (C1, C3, C5), and Daubechies (D4, D12, D20), were used. The level of decomposition also significantly impacts the reconstructed spectrum. A high level of decomposition may distort the spectral peaks. The optimal level of decompositions should be investigated on the simulated spectra by calculating the RMS between the reconstructed and the real spectrum at each level of decomposition.

Experimental spectra

An AHPGe detector coupled with an analog digital converter (ADC), high voltage 5000 V, negative polarity and relative efficiency 70 %, was used to detect different sources of gamma spectra. Genie 2000 software, (Canberra Industries, Meriden, USA), with an analyzer cart recorded the intensity of the incident and the transmitted gamma rays. The detected spectra can be translated to ASCII and processed with custom-made programs based on ROOT. Automatic pulse-shaping and pole-zero correction settings were used and the energy scale calibrated by ^{241}Am, ^{57}Co, ^{137}Cs, and ^{60}Co radioactive sources. The resolution of the system was 1.9 keV at 1332.5 keV gamma peaks of a ^{60}Co point source kept at a distance of 10 cm in front of the detector face. The whole system was housed in a lead shielding of thicknesses varying up to 12 cm in order to suppress the background noise effect. Different kinds of spectra resulted from the denoising of background and mixed calibration spectra [13].

RESULTS AND DISCUSSION

Although our study has focused on testing the DWT method for denoising the gamma-ray spectrum, a comparison with other familiar methods was applied as well. The choice of methods to be compared is probably the most questionable point of the comparison presented. We have chosen two different advanced methods for denoising: that of the Savitzky-Golay and Fourier transform.

The Savitzky-Golay algorithm is an indirect filter because it is carried out in time, rather than in the frequency domain. It is based on local least-squares polynomial approximation and the subsequent evaluation of the resulting polynomial at a single point within the approximation interval. The Savitzky-Golay algorithm has shown that a set of integers (a_{-n}, $a_{-(n-1)}$, …, a_{n-1}, a_n) could be derived and used as weighting coefficients for carrying out the smoothing operation. The use of these weighting coefficients, known as convolution integers, turns out to be exactly equivalent to the fitting of data to the above described polynomial, while computationally more effective and much faster. Therefore, the smoothed data point Y_k by the Savitzky-Golay algorithm is given by the following eq. [14]

$$Y_\mathrm{k} = \frac{\sum_{i=1}^{n} a_i Y_{k+i}}{\sum_{i=1}^{n} a_\mathrm{i}} \qquad (18)$$

Many sets of convolution integers can be applied, depending on filter width and polynomial degree. Sets of convolution integers can be used to obtain directly, instead of the smoothed signal, its 1$^{\mathrm{st}}$, 2$^{\mathrm{nd}}$,…, m^{th} order derivative, therefore the Savitzky-Golay algorithm is very useful for calculating the derivatives of noisy signals consisting of discrete and equidistant points.

The Fourier transform is a tool for processing signals in both the time and the frequency domain. It is based on the idea of decomposing periodic signals into their harmonic components. Many years after Fourier had discovered it, a new algorithm called the fast Fourier transform (FFT) was developed and became even more popular. Fourier filtering methods are based on decomposing a signal into its frequency components. By suppressing the high frequency components, one can achieve a denoising effect. Both DWT and FFT transforms are invertible, and both sets of basis functions are orthogonal. However, Fourier basis functions are completely localized in the frequency domain and have an infinite extent in the time domain, whereas the wavelet basis functions are dually localized in both the time and frequency domain [15].

Testing the wavelet for denoising spectra starts with the chosen wavelet function. The performance of denoising was evaluated by computing the root mean square (RMS) ratio between the reconstructed spectrum and the ideal one (tab. 1). It can be seen that the Haar wavelet performs best of all filters. The good results that were obtained with this filter are probably due to the stair-step features of the Haar wavelet which contributes to the major part of the spectrum. The optimal level of decomposition for the Haar wavelet on the simulated spectra was investigated by calculating the RMS between the reconstructed and the real spectrum at each level of decomposition, as presented in tab. 2. Results showed that the spectrum should be decomposed until 24 approximation coefficients remain. This corresponds to six levels of decomposition. When a higher level of decomposition is used, strong artifacts are present around the peaks. The denoising of gamma spectra using the Haar wavelet and the six levels of decomposition was tested on different types of spectra. The first test was applied to a simulated spectrum with single and overlapped peaks surrounded by high levels of noise. The denoising of the simulated signals using the DFT was compared with Savitzky-Golay and FFT denoising, as shown in fig. 2. It is clear that the denoising of the spectrum by a DWT filter is more efficient and that it removes most of the noise. Figure 3 shows a comparison of DFT, FFT, and Savitzky-Golay filters for the removal of noise without a simulated spectrum containing single and overlapped peaks with high noise. It is clear that DFT is more effective in removing noise than other filters without causing any damage. FFT and Savitzky-Golay filters remove noise along the entire spectrum, but with damage to the ratio of big peak shapes.

Table 1. Influence of the wavelet type on the reconstructed spectrum

Wavelet	H	D4	D12	D20	C1	C3	C5
RMS	0.022	0.023	0.033	0.005	0.034	0.031	0.039

Table 2. Influence of decomposition levels on reconstructed signals

Level	RMS			
	Single	Overlapped	Background	Marinelli
1	0.0441	0.0680	0.0587	0.0588
2	0.0294	0.0660	0.0503	0.0426
3	0.0130	0.0638	0.0499	0.0346
4	0.0182	0.0604	0.0454	0.0288
5	0.0103	0.0579	0.0455	0.0250
6	0.0040	0.0570	0.0406	0.0230
7	0.0053	0.0666	0.0513	0.0242
8	0.0056	0.0712	0.0581	0.0246
9	0.0066	0.0726	0.0600	0.0289

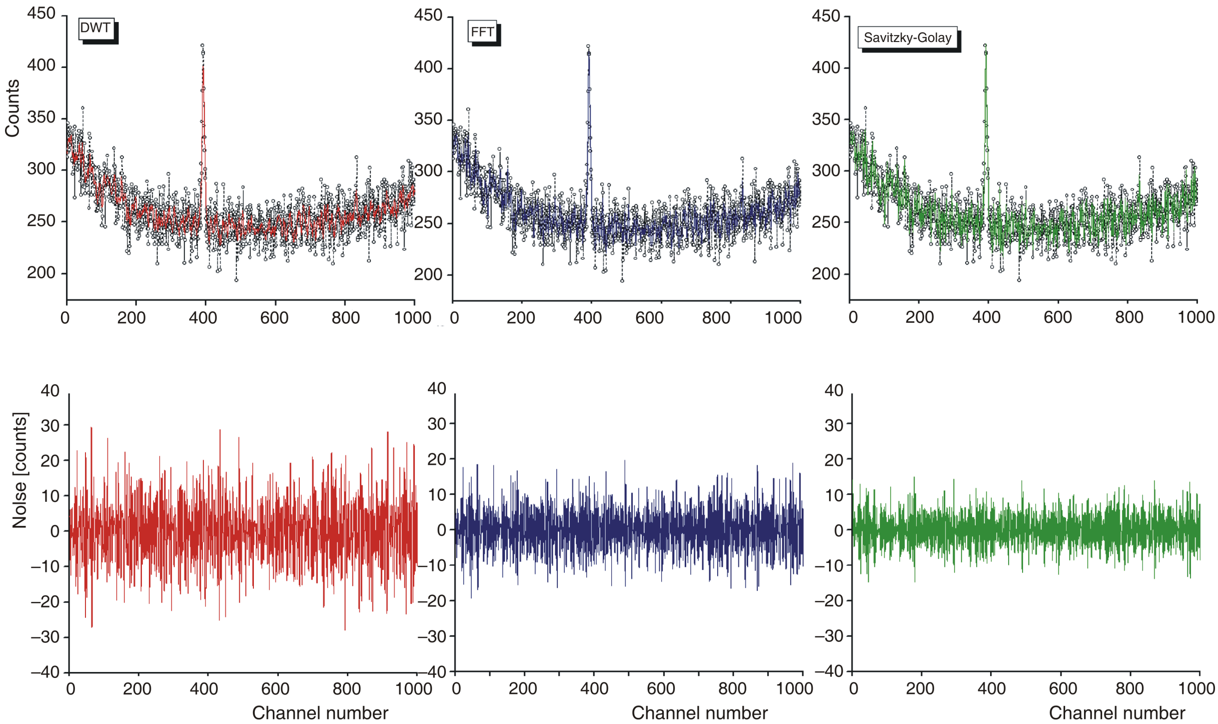

Figure 2. Comparison of DFT, FFT, and Savitzky-Golay of a simulated spectrum containing a single peak with high noise

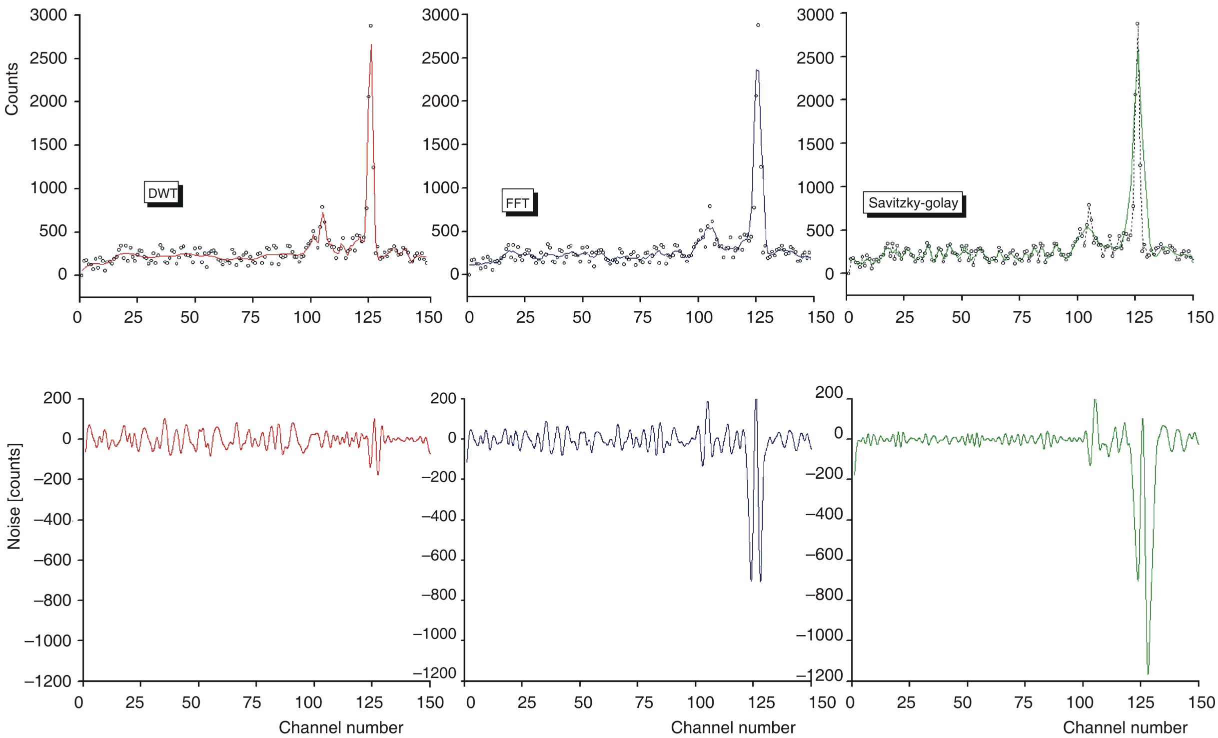

Figure 3. Comparison of DFT, FFT, and Savitzky-Golay of a simulated spectrum containing a single peak and overlapped peaks with high noise

A background spectrum of the laboratory was acquired for the purpose of comparing the denoising spectrum with the effectiveness of the lead shielding. The main lines that can be observed are those of ^{222}Rn daughters (^{214}Pb and ^{214}Bi), ^{40}K, ^{208}Tl, as well as the annihilation line at 511 keV, caused by cosmic rays. The comparison of denoising techniques of the detected background spectra is shown in fig. 4. It is clear that the highest integral areas from the main gamma transitions were noted under the peak at 351.9 keV (^{214}Pb), followed by those under peaks at annihilation 511.0 keV, 609.3 (^{214}Bi), and 1460.8 keV (^{40}K). The

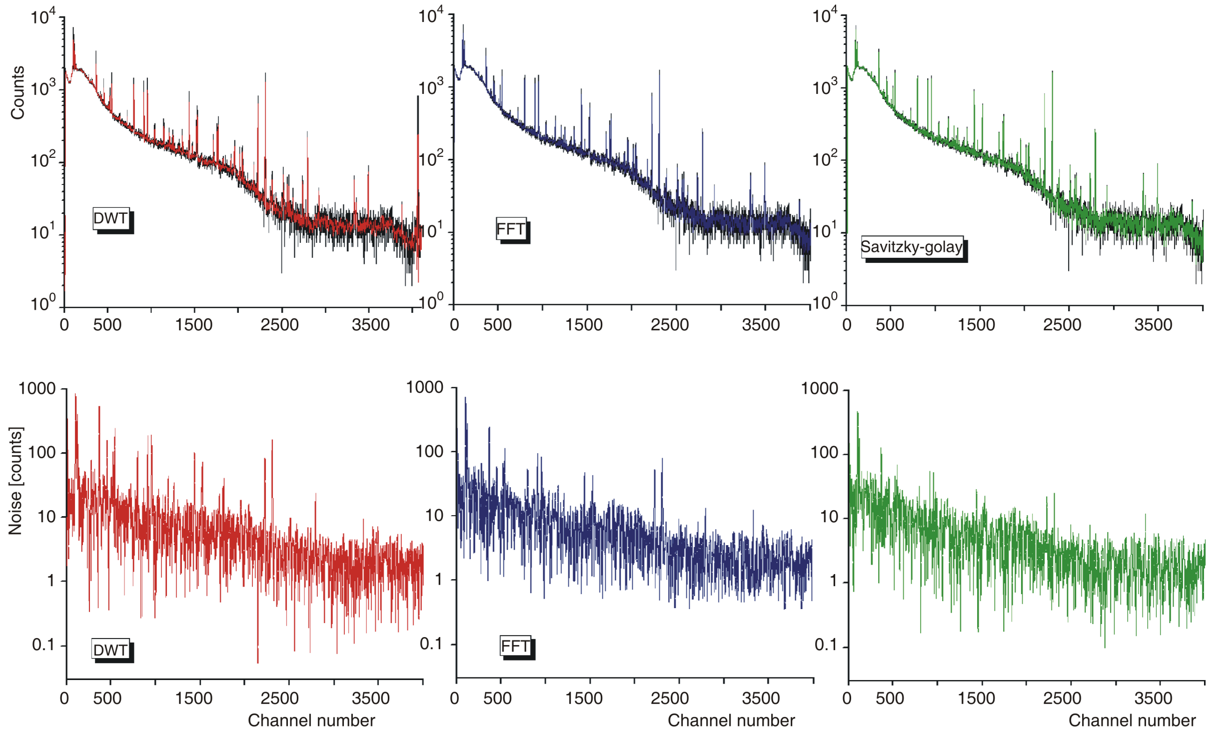

Figure 4. Comparison of DFT, FFT, and Savitzky-Golay of the detected background spectrum of an unshielded HPGe detector

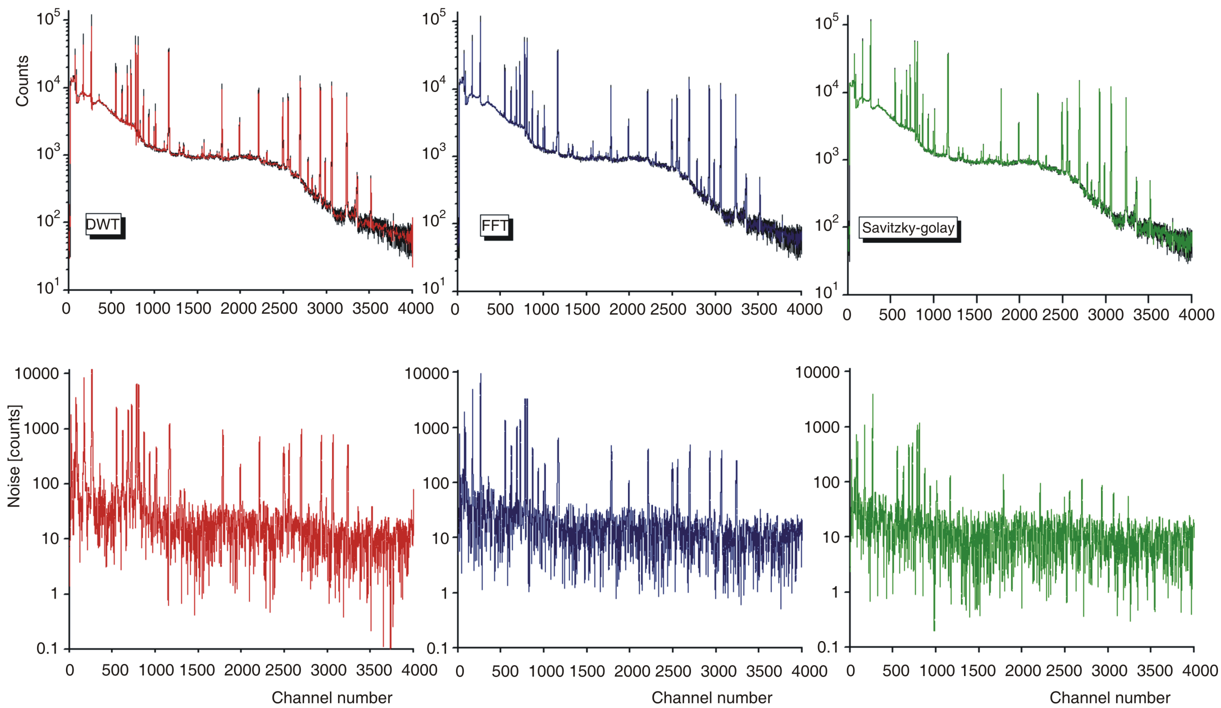

Figure 5. Comparison of DFT, TTF, and Savitzky-Golay of the detected spectrum of a Marinelli beaker with different gamma ray source

two most intense gamma transitions from the thorium series *i. e.*, 911.6keV (^{228}Ac) and 583.1 keV (^{208}Tl), are characterized by reduced areas compared to the lines of 351.9, 609.3, and 1460.8 keV.

Finally, the DWT filter was applied to the detected spectrum of a 500 mL Marinelli beaker filled with resin of a density of 1.15 g/cm^3 containing a mixture of isotopes (^{241}Am, ^{109}Cd, ^{139}Ce, ^{57}Co, ^{134}Cs,^{137}Cs, ^{203}Hg, ^{54}Mn,^{113}Sn, ^{88}Y, and ^{65}Zn) with different activities. The detected gamma spectrum is shown in fig. 5. It is clear that DWT represents the best filtering denoising technique for the detected gamma spectrum. Our study has also shown that, in comparison to FFT and Savitzky-Golay, the denoising of the spectrum by DWT is much more efficient.

CONCLUSION

The discrete wavelet transform is shown to be useful in denoising the experimental gamma spectrum as well as the simulated spectra. Two simple approaches for filtering the transformed vector have been systematically investigated. The DWT results were compared to results obtained using the Savitzky-Golay algorithm, as well as those pertaining to the FFT filtering algorithm. Wavelet filtering algorithms have been shown to denoise data to an extent superior to other methods.

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МОДЕРНА МАТЕМАТИЧКА МЕТОДА ЗА ФИЛТРИРАЊЕ ШУМА У ЕКСПЕРИМЕНТИМА НИСКОГ ОДБРОЈА

У раду је размотрена нова примена нумеричке и функционалне анализе заснована на дискретној таласној трансформацији и описан је математички поступак за побољшање сигнала и одстрањивање шума. Добијени резултати показују да је метода примењена на разноврсним гама спектрима у предности над другим техникама.

Кључне речи: шум, сигнал, дискретна таласна трансформација